Trees on Tracks*

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1. Introduction

Simultaneous embedding of planar graphs is related to the problems of graph thickness and geometric thickness. Techniques for simultaneous embedding of cycles have been used to show that the degree-4 graphs have geometric thickness at most two [3]. Simultaneous embedding techniques are also useful in visualization of graphs that evolve through time.

The notion of simultaneous embedding is related to that of graph thickness. Two vertex-labeled planar graphs on n vertices can be simultaneously embedded if there exist a labeled point set of size n such that each of the graphs can be realized on that point set (using the vertex-point mapping defined by the labels) with straight-line edge segments and without crossings. For example, any two paths can be simultaneously embedded, while there exist pairs of outerplanar graphs that do not have a simultaneous embedding.

In this paper we present new results about embedding labeled trees and outerplanar graphs on labeled tracks, as well as related results on simultaneous embedding of tree-path pairs. In particular, we show that labeled trees cannot be embedded on labeled parallel straight-line tracks, but they can be embedded on labeled concentric circular tracks; see Fig. 1. The results generalize to outerplanar graphs as well. We also show that tree-path pairs can be simultaneously embedded when edges of the path are represented by circular arcs. Finally, we show how to embed a straight-line tree and a path with $O(\log n)$-bends per edge, where n is the number of vertices.

1.1. Related Work

The existence of straight-line, crossing-free drawings for a single planar graphs is well known [5]. The existence of simultaneous geometric embeddings for pairs of paths, cycles, and caterpillars is shown in [1]. Counter-examples for pairs of general planar graphs, pairs of outer-planar graphs, and triples of paths are also presented there. It it not known whether tree-tree or tree-path pairs allow simultaneous geometric embeddings. If the straight-line edge condition is relaxed, it is known how to embed tree-path pairs using one bend per tree edge and how to embed tree-tree pairs using at most 3 bends per edge [4].

A related problem is the problem of *graph thickness* [7], defined as the minimum number of planar subgraphs into which the edges of the graph can be partitioned into. *Geometric thickness* is a version of the thickness problem where the edges are required to be straight-line segments [2]. Thus, if two graphs have a simultaneous geometric embedding, then their union has geometric thickness two. Similarly, the union of any two planar graphs has graph thickness two. Simultaneous geometric embedding techniques are used to show that degree-four graphs have geometric thickness two [3].

Simultaneous drawing of multiple graphs is also related to the problem of embedding planar graphs on a fixed set of points in the plane. Several variations of this problem have been studied. If the mapping between the vertices V and the points P is not fixed, then the graph can be drawn without crossings using two bends per edge in polynomial time [6]. However, if the mapping between V and P is fixed, then $O(n)$ bends per edge are necessary to guarantee planarity, where n is the number of vertices in the graph [8].

1.2. Our Contributions

We begin with results on track embeddability. Given a set of labeled parallel lines (tracks) L_i, $1 \leq i \leq n$ and a tree $T = (V, E)$ with n vertices labeled with the numbers 1 though n, it is not always possible to obtain a straight-line crossings-free drawing of T such that vertex v_i is on track L_i; see Fig. 1(a). However, if the tracks are concentric circles, such drawings are always possible and we describe a linear time algorithm for obtaining such drawings; see Fig. 1(b). The algorithm easily generalizes to outerplanar graphs as well. Thus, parallel line tracks do not allow tree or outerplanar embeddings on predetermined tracks, while circular tracks do. Tracks defined by circular arcs, stairs, sin-waves also suffice; see Fig. 2.

Our motivation for the problem of track embeddings comes from two open problems in simultaneous geometric embedding. Formally, in the problem of simultane-

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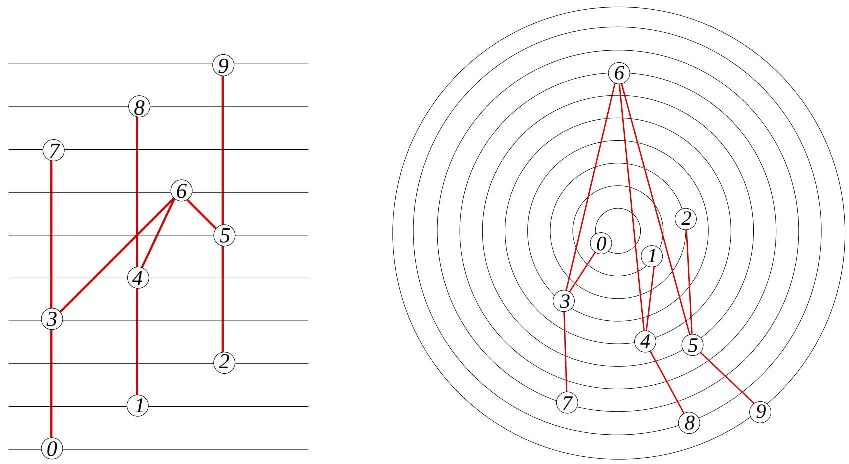

Figure 1: A tree that cannot be drawn on line tracks without crossings but which can be drawn on circular tracks.

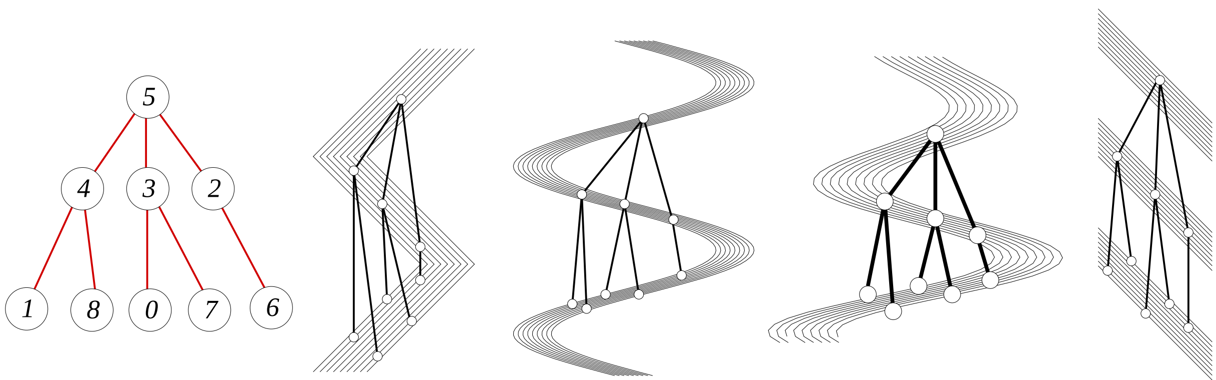

Figure 2: A tree drawn on various staircases: (a) staircase; (b) $\sin(x)$; (c) $x\sin(x)$; (d) $x - \lfloor x \rfloor$.

ous geometric embedding we are given two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ and we would like to find plane straight-line drawings D_1 and D_2 such that for all vertices $v \in V$ the location of the corresponding vertices in D_1 and D_2 is the same (i.e., $D_i(v) = D_j(v)$). While path-path, cycle-cycle, caterpillar-caterpillar pairs can be simultaneously embedded, it is not known whether tree-tree or tree-path pairs have such embeddings.

The circular track layout of trees and outerplanar graphs can be used to obtain simultaneous embeddings of tree-path pairs so that the tree edges are straight-line and crossings-free and the path edges are crossings-free circular-arc segments. Moreover, the staircase layout of trees can be used to obtain simultaneous embeddings of tree-path pairs so that the tree edges are straight-line and crossings-free and the path edges are crossings-free and have at most $\log n$ bends per edge.

2. References

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